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INTEGRAL EQUATION FORMULATION OF THE BOUNDARY

VALUE PROBLEMS OF ELASTICITY

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ABSTRACT

The three fundamental boundary value problems of the classical theory of linear elasticity have been formulated in a manner that permits maximum exploitation of numerical techniques and modern high speed digital computers. Both the static and dynamic cases have been included as well as arbitrary body forces. The extension to problems involving more than one media has been given and in addition no restrictions are placed on the boundary shape.

Introduction

In the classical theory of linear elasticity there are two classes of problems: dynamic and static. In each of these classes there are three fundamental boundary value problems which are labeled the first, second and third or mixed boundary value problems respectively. The first problem consists in determining the displacements in an elastic body when the stresses are specified on the boundary and the second problem is the determination of the displacements throughout an elastic body when the displacements are given on the boundary. The third or mixed boundary value problem requires the determination of the displacements in an elastic body if the stresses are prescribed over part of the boundary and the displacements are given over the remainder.[1], [2],[3],[4]*

Many important problems in elasticity may only be solved by employing numerical methods. In order to permit maximum use of numerical techniques it is desirable that all three boundary value problems, for both the static and dynamic cases, be formulated in a similar manner. Such a formulation can be achieved by employing methods analogous to those used in potential theory. In this paper the complete formulation of these problems will be given. This formulation has been used to obtain numerical results to some specific problems. These results have been given in reference (5) and will not be reproduced here.

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*Superscript numbers in parenthesis refer to references in bibliography.

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Some General Considerations

The dynamic equations of classical linear elasticity may be written in terms of the displacement vector, $\vec{u}$, as

$$(\lambda + 2\mu)\nabla(\nabla\cdot\vec{u}) - \mu\nabla\times\nabla\times\vec{u} + \rho\vec{F} = \rho\vec{u}_{tt} \qquad (1)$$

where λ and μ are the Lame constants, $\vec{F}$ is the body force vector and lower case subscript letters indicate partial differentiation.[6] $\vec{u}$ is the displacement vector with components u, v, w in the x, y, z directions respectively. The static equilibrium equations are obtained by setting the inertia force terms equal to zero in the above equations. They are

$$(\lambda + 2\mu)\nabla(\nabla\cdot\vec{u}) - \mu\nabla\times\nabla\times\vec{u} + \rho\vec{F} = 0 \qquad (2)$$

Using the fact that a vector function may be decomposed into a solenoidal and an irrotational part, the displacement vector may be written as

$$\vec{u} = \nabla\phi + \nabla\times\vec{\Psi} \qquad (3)$$

where $\nabla\cdot\vec{\Psi} = 0$. The solenoidal part, $\nabla\times\vec{\Psi}$, corresponds to a rotation and the irrotational part, $\nabla\phi$, to a dilatation. The body force, $\vec{F}$, may also be written as

$$\vec{F} = \nabla f + \nabla\times\vec{A}$$

where

$$f = -\frac{1}{4\pi}\iiint \vec{F}\cdot\nabla\left(\frac{1}{R}\right)d\tau$$

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and
$$\vec{A} = \frac{1}{4\pi} \iiint \vec{F} \times \nabla\left(\frac{1}{\lambda}\right) d\tau$$

providing that $\vec{F}$ is uniformly continuous and vanishes at infinity in a prescribed manner.

Substitution of Equation (3) into Equation (1) and employing the vector identities

$$\nabla \times \nabla \phi = 0 \quad , \qquad \nabla \cdot \nabla \times \vec{\Psi} = 0$$

and

$$\nabla \times \nabla \times \vec{\Psi} = \nabla(\nabla \cdot \vec{\Psi}) - \nabla^2 \vec{\Psi}$$

shows that if ϕ and $\vec{\Psi}$ satisfy the inhomogeneous wave equations

$$\nabla^2 \phi + f_1 = \frac{1}{C_1^2} \phi_{tt} \qquad \text{and} \quad \nabla^2 \vec{\Psi} + \vec{A_2} = \frac{1}{C_2^2} \vec{\Psi}_{tt} \qquad (4)$$

where
$$f_1 = \frac{f}{\lambda + 2\mu} \quad , \qquad \vec{A_2} = \frac{\vec{A}}{\mu}$$

$$C_1^2 = \frac{\lambda + 2\mu}{\rho} \quad , \qquad C_2^2 = \frac{\mu}{\rho}$$

then the displacement vector as given by Equation (3) satisfies Equation (1). Similarly, the representation (3), when substituted into Equation (2) shows that if ϕ and $\vec{\Psi}$ satisfy the Poisson equations

$$\nabla^2 \phi + f_1 = 0 \qquad \text{and} \quad \nabla^2 \vec{\Psi} + \vec{A_2} = 0 \qquad (5)$$

then the displacement vector, $\vec{u}$, satisfies the static equilibrium equations.

Recalling the integral representation of the solution of the inhomogeneous wave equation, the functions ϕ and $\vec{\Psi}$ may

be written as

$$\phi(P,t) = \frac{1}{4\pi} \iiint [f_1] \left(\frac{1}{r}\right) d\tau$$

$$- \frac{1}{4\pi} \iint \left\{ [\phi]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{\partial \phi}{\partial n}\right] - \frac{1}{c_1 r}\frac{\partial r}{\partial n}\left[\frac{\partial \phi}{\partial t}\right] \right\} d\sigma \qquad (6)$$

and

$$\vec{\Psi}(P,t) = \frac{1}{4\pi} \iiint [\vec{A}_1] \left(\frac{1}{r}\right) d\tau$$

$$- \frac{1}{4\pi} \iint \left\{ [\vec{\Psi}]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{\partial \vec{\Psi}}{\partial n}\right] - \frac{1}{c_1 r}\frac{\partial r}{\partial n}\left[\frac{\partial \vec{\Psi}}{\partial t}\right] \right\} d\sigma \qquad (7)$$

where $\frac{\partial}{\partial n}$ denotes partial differentiation in the direction of
the exterior normal to σ, r is the distance from the observa-
tion point P to a point P_0 on σ and σ is the closed surface
enclosing the volume τ .[7] [F] means that the function F(x,y,z,t)
is to be evaluated at the retarded time t' = t - r/c. Equations
(6) and (7) are Kirchoff's solutions of the inhomogeneous wave
equations in terms of retarded potentials. The corresponding
integral representations of the solutions of the Poisson equations,
(5) are [8]

$$\phi(P) = \frac{1}{4\pi} \iiint \frac{f_1}{r} d\tau + \frac{1}{4\pi} \iint \left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right] d\sigma \qquad (8)$$

and

$$\vec{\Psi}(P) = \frac{1}{4\pi} \iiint \frac{\vec{A}_2}{r} d\tau + \frac{1}{4\pi} \iint \left[\vec{\Psi}\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \vec{\Psi}}{\partial n} \right] d\sigma \qquad (9)$$

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If body forces are not present then equations (4) and (5) are
replaced by the homogeneous wave equations and Laplaces' equa-
tions respectively and the volume integrals in the above integrals
vanish.

When a harmonic time dependence of the form $e^{-i\omega t}$ is
assumed, the displacement potentials must satisfy the following
set of inhomogeneous Helmholtz equations

$$\nabla^2 \phi + f_1 + k_1^2 \phi = 0 \quad \text{and} \quad \nabla^2 \vec{\psi} + \vec{A}_2 + k_2^2 \vec{\psi} = 0 \qquad (10)$$

where

$$k_1^2 = \frac{\omega^2}{C_1^2} \qquad \text{and} \qquad k_2^2 = \frac{\omega^2}{C_2^2} \quad .$$

The integral representations for ϕ and $\vec{\psi}$ are[9]

$$\phi(P) = \frac{1}{4\pi} \iiint f_1 \frac{e^{ik_1 R}}{R} \, d\tau$$
$$+ \frac{1}{4\pi} \iint \left\{ \phi \frac{\partial}{\partial n}\left(\frac{e^{ik_1 R}}{R}\right) - \frac{e^{ik_1 R}}{R} \frac{\partial \phi}{\partial n} \right\} d\sigma \qquad (11)$$

and

$$\vec{\psi}(P) = \frac{1}{4\pi} \iiint \vec{A}_2 \frac{e^{ik_2 R}}{R} \, d\tau$$
$$+ \frac{1}{4\pi} \iint \left\{ \vec{\psi} \frac{\partial}{\partial n}\left(\frac{e^{ik_2 R}}{R}\right) - \frac{e^{ik_2 R}}{R} \frac{\partial \vec{\psi}}{\partial n} \right\} d\sigma \quad . \qquad (12)$$

As before, the volume integrals in each of the above vanish if there are no body forces present.

In differential geometry it is shown that an arbitrary surface in a three-dimensional space may be characterized by the parametric representation

$$\vec{\chi} = \vec{\chi}(\alpha, \beta) \tag{13}$$

where α and β are two independent parameters. If the lines of curvature at a point of the surface are chosen as the parametric representation of the surface then a local orthogonal coordinate system may be constructed at the point. This is always possible for a sufficiently smooth surface.[10] The coordinate system consists of the two lines of curvature and the normal to the tangent plane at the point. The unit normal is given by

$$\hat{n} = \frac{\vec{\chi}_\alpha \times \vec{\chi}_\beta}{|\vec{\chi}_\alpha \times \vec{\chi}_\beta|} \tag{13}$$

It is assumed that the surface is sufficiently smooth so that the parametric representation as given by Equation (13) possesses as many derivatives as may be required in the subsequent development.

In the next section it will be shown that by employing the integral representations of the functions ϕ and $\vec{\psi}$ and imposing the proper boundary conditions that a set of integral equations may be derived whose solutions are the desired surface potentials from which the displacement and stress fields may be calculated by integration.

Formulation of the Integral Equations

The first problem to be considered is the second fundamental boundary value problem of elasticity. Let $\vec{D}$ denote the prescribed surface displacement vector and $\vec{u}$ the displacement vector obtained from the displacement potentials. On σ the normal and tangential components of the displacement must be equal and hence on σ the following boundary conditions hold:

$$\vec{u} \cdot \hat{n} = \vec{D} \cdot \hat{n} \qquad \text{and} \qquad \vec{u} \times \hat{n} = \vec{D} \times \hat{n} \quad . \qquad (14)$$

In terms of the displacement potentials the equations are

$$\nabla \phi \cdot \hat{n} + \nabla \times \vec{\Psi} \cdot \hat{n} = \vec{D} \cdot \hat{n} \qquad (15)$$

and

$$\nabla \phi \times \hat{n} + \nabla \times \vec{\Psi} \times \hat{n} = \vec{D} \times \hat{n} \qquad (16)$$

In a previous paper,[5] the following vector relationships were derived:

$$H \nabla \phi \times \hat{n} = \vec{\chi}_\alpha \phi_\beta - \vec{\chi}_\beta \phi_\alpha \, , \quad H \nabla \vec{\Psi} \cdot \hat{n} = \vec{\chi}_\alpha \cdot \vec{\Psi}_\beta - \vec{\chi}_\beta \cdot \vec{\Psi}_\alpha$$

and

$$H \nabla \times \vec{\Psi} \times \hat{n} = H \frac{\partial \vec{\Psi}}{\partial n} + \vec{\chi}_\alpha \times \vec{\Psi}_\beta - \vec{\chi}_\beta \times \vec{\Psi}_\alpha$$

where the fact that $\vec{\Psi}$ is divergenceless has been used and

$$H = \sqrt{EG - F^2} \qquad \text{with} \qquad E = \vec{\chi}_\alpha \cdot \vec{\chi}_\alpha$$

$$G = \vec{\chi}_\beta \cdot \vec{\chi}_\beta \qquad \text{and} \qquad F = \vec{\chi}_\alpha \cdot \vec{\chi}_\beta = 0$$

since the local coordinate system is orthogonal. Application of these vector relationships to Equations (15) and (16) gives

$$H\phi_n + \vec{\chi}_\alpha \cdot \vec{\Psi}_\beta - \vec{\chi}_\beta \cdot \vec{\Psi}_\alpha = H\vec{D} \cdot \hat{n} \qquad (17)$$

and

$$H\vec{\Psi}_n + \vec{\chi}_\alpha \phi_\beta - \vec{\chi}_\beta \phi_\alpha + \vec{\chi}_\alpha \times \vec{\Psi}_\beta - \vec{\chi}_\beta \times \vec{\Psi}_\alpha = H\vec{D} \times \hat{n} \ . \qquad (18)$$

Equations (17) and (18) may be solved for the surface values of ϕ_n and $\vec{\Psi}_n$ in terms of the tangential derivatives of ϕ and $\vec{\Psi}$. If the prescribed surface displacement vector is time dependent the values of ϕ_n and $\vec{\Psi}_n$ are substituted into Equations (6) and (7) to give

$$\phi(\vec{p},t) = \frac{1}{4\pi} \int\int\int [f_1]\left(\frac{1}{\Lambda}\right) d\tau$$

$$-\frac{1}{4\pi}\int\int \left\{ H[\phi]\frac{\partial}{\partial n}\left(\frac{1}{\Lambda}\right) + \frac{1}{\Lambda}\left[\vec{\chi}_\alpha \cdot \vec{\Psi}_\beta - \vec{\chi}_\beta \vec{\Psi}_\alpha\right]\right.$$

$$\left. -\frac{1}{C_1 \Lambda}\frac{\partial}{\partial n}\left[\frac{\partial \phi}{\partial t}\right]\right\} d\alpha \, d\beta \qquad (19)$$

$$-\frac{1}{4\pi}\int\int H[\vec{D} \cdot \hat{n}]\left(\frac{1}{\Lambda}\right) d\alpha \, d\beta$$

and

$$\vec{\Psi}(P,t) = \frac{1}{4\pi} \iiint [\vec{A}_2]\left(\frac{1}{\hbar}\right) d\tau$$

$$-\frac{1}{4\pi} \iint \left\{ H[\vec{\Psi}]\frac{\partial}{\partial n}\left(\frac{1}{\hbar}\right) + \frac{1}{\hbar}\left[\vec{\chi}_\alpha \phi_\beta - \vec{\chi}_\beta \phi_\alpha \right.\right.$$

$$\left.\left. + \vec{\chi}_\alpha \times \vec{\Psi}_\beta - \vec{\chi}_\beta \times \vec{\Psi}_\alpha \right] - \frac{1}{C_2 \hbar}\left[\frac{\partial \vec{\Psi}}{\partial t}\right]\right\} d\alpha\, d\beta$$

$$-\frac{1}{4\pi} \iint H[\vec{D} \times \hat{n}]\left(\frac{1}{\hbar}\right) d\alpha\, d\beta \quad .$$

(20)

Allowing the observation point P to approach the boundary and
interpreting the integrals in the sense of their limiting values
results in a set of four simultaneous integral equations for
the determination of the surface values of the displacement
potentials. Solution of these equations for the surface poten-
tials enables the field values to be calculated by integration.
This is the integral equation formulation of the second funda-
mental boundary value problem of dynamic elasticity. If the
prescribed surface displacement vector is not time dependent the
problem is a static one and equation (2) is the governing
differential equation. Consequently, the surface values of ϕ_n
and $\vec{\Psi}_n$ obtained from Equations (17) and (18) are substituted into
Equations (8) and (9) to give the required integral representations
for ϕ and $\vec{\Psi}$. By interpreting these integrals in the sense
of their limiting values the integral equation formulation of the

second fundamental boundary value problem of static elasticity
is obtained. If a harmonic time dependence is assumed for both
the prescribed displacement vector and the body force, the
values of ϕ_n and $\vec{\psi}_n$ are substituted into the integral
representations of the inhomogeneous Helmholtz equations. The
limiting values of the resulting integrals gives a set of simul-
taneous integral equations which are equivalent to those obtained
from the integral equation formulation of the scattering of steady
harmonic elastic waves from rigid surfaces of arbitrary shape.

The next problem to be considered is the first fundamental
problem of elasticity. Let $\vec{T}$ denote the stress vector prescribed
on the boundary σ and $\vec{S}$ the stress in the body acting on a plane
with direction cosines l, m, n. Then

$$\vec{S} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k} \tag{21}$$

where

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \sigma^{xx} & \sigma^{xy} & \sigma^{xz} \\ \sigma^{xy} & \sigma^{yy} & \sigma^{yz} \\ \sigma^{xz} & \sigma^{yz} & \sigma^{zz} \end{pmatrix} \begin{pmatrix} \ell \\ m \\ n \end{pmatrix} \tag{22}$$

The stress-strain relations in usual notation[11] are

$$\sigma^{xx} = \lambda\theta + 2\mu u_x \qquad \sigma^{xy} = \mu(u_x + v_y)$$

$$\sigma^{yy} = \lambda\theta + 2\mu v_y \qquad \sigma^{yz} = \mu(v_z + w_y) \tag{23}$$

$$\sigma^{zz} = \lambda\theta + 2\mu w_z \qquad \sigma^{xz} = \mu(w_x + u_z)$$

where the dilatation, θ, is given by

$$\theta = \nabla \cdot \vec{u} \quad . \tag{24}$$

The boundary conditions require that the normal and tangential components of the stress at the boundary must be equal, i.e.

$$\vec{S} \cdot \hat{n} = \vec{T} \cdot \hat{n} \tag{25}$$

and

$$\vec{S} \times \hat{n} = \vec{T} \times \hat{n} \qquad \text{on} \quad \sigma \quad . \tag{26}$$

Substitution of Equations (21) and (24) into Equation (25) gives

$$\sigma^{xx} \ell^2 + \sigma^{yy} m^2 + \sigma^{zz} n^2$$

$$+ 2\sigma^{xy} \ell m + 2\sigma^{yz} mn + 2\sigma^{xz} \ell n = \vec{T} \cdot \hat{n} \tag{27}$$

Relating the stresses to the displacement potentials by using Equations (3) and (25) and substituting into the above equation yields

$$C_1^2 H^2 \phi_{nn} + (C_1^2 - 2C_2^2)(\phi_{\alpha\alpha} G + \phi_{\alpha\alpha} E)$$

$$+ 2 H C_2^2 (\vec{\chi}_\beta \cdot \vec{\psi}_{n\alpha} - \vec{\chi}_\alpha \cdot \vec{\psi}_{n\beta}) = \frac{\vec{T} \cdot \hat{n}}{\rho} \tag{28}$$

In a similar manner Equation (26) may be written in terms of the displacement potentials to give

$$C_2^2 \Big\{ H^2 \vec{\psi}_{nn} + H \big(2\vec{\chi}_\alpha \phi_{n\beta} - 2\vec{\chi}_\beta \phi_{n\alpha} + \vec{\chi}_\alpha \times \vec{\psi}_{n\beta} - \vec{\chi}_\beta \times \vec{\psi}_{n\alpha} \big)$$

$$- \Big[\vec{\chi}_\beta (\vec{\chi}_\beta \cdot \vec{\psi}_{\alpha\alpha}) - \vec{\chi}_\alpha (\vec{\chi}_\beta \cdot \vec{\psi}_{\alpha\beta}) + \vec{\chi}_\alpha (\vec{\chi}_\alpha \cdot \vec{\psi}_{\beta\beta})$$

$$- \vec{\chi}_\beta (\vec{\chi}_\alpha \cdot \vec{\psi}_{\alpha\beta}) \Big] \Big\} = \frac{\vec{T} \times \hat{n}}{\rho} \tag{29}$$

The algebra necessary to obtain equations (28) and (29) is quite lengthy and is given in a previous paper.[5] If the prescribed surface stress is time dependent, Equation (1) is the governing differential equation and the displacement potentials satisfy Equations (4). Since the local coordinate system is orthogonal,

$$\nabla^2 f = \frac{1}{\sqrt{E}}\left[f_\alpha \frac{\partial}{\partial \alpha}\left(\frac{1}{\sqrt{E}}\right) + \frac{f_{\alpha\alpha}}{\sqrt{E}}\right]$$

$$+ \frac{1}{\sqrt{G}}\left[f_\beta \frac{\partial}{\partial \beta}\left(\frac{1}{\sqrt{G}}\right) + \frac{f_{\beta\beta}}{\sqrt{G}}\right] + f_{nn} \tag{30}$$

and Equations (4) may be combined with Equations (28) and (29) to eliminate ϕ_{nn} and $\overrightarrow{\Psi}_{nn}$ giving:

$$H^2\left\{ \phi_{tt} - \frac{C_1^2}{\sqrt{E}}\left[\phi_\alpha \frac{\partial}{\partial \alpha}\left(\frac{1}{\sqrt{E}}\right) + \frac{\phi_{\alpha\alpha}}{\sqrt{E}}\right] \right.$$

$$\left. + \frac{C_1^2}{\sqrt{G}}\left[\phi_\beta \frac{\partial}{\partial \beta}\left(\frac{1}{\sqrt{G}}\right) + \frac{\phi_{\beta\beta}}{\sqrt{G}}\right] - C_1^2 f_1 \right\} + (C_1^2 - 2C_2^2)$$

$$(\phi_{\alpha\alpha} G + \phi_{\beta\beta} E) + 2 H C_2^2 \left(\vec{\chi}_\beta \cdot \overrightarrow{\Psi}_{n\alpha} - \vec{\chi}_\alpha \cdot \overrightarrow{\Psi}_{n\beta}\right) = \frac{\overrightarrow{T}\cdot\hat{n}}{\rho} \tag{31}$$

and

$$H^2\left\{ \overrightarrow{\Psi}_{tt}\left(\frac{1}{C_2^2}\right) - \frac{1}{\sqrt{E}}\left[\overrightarrow{\Psi}_\alpha \frac{\partial}{\partial \alpha}\left(\frac{1}{\sqrt{E}}\right) + \frac{\overrightarrow{\Psi}_{\alpha\alpha}}{\sqrt{E}}\right] \right.$$

$$\left. - \frac{1}{\sqrt{G}}\left[\overrightarrow{\Psi}_\beta \frac{\partial}{\partial \beta}\left(\frac{1}{\sqrt{G}}\right) + \frac{\overrightarrow{\Psi}_{\beta\beta}}{\sqrt{G}}\right] - \overrightarrow{A}_2 \right\} + H\left(2\vec{\chi}_\alpha \phi_{n\beta} - 2\vec{\chi}_\beta \phi_{n\alpha}\right.$$

$$\left. + \vec{\chi}_\alpha \times \overrightarrow{\Psi}_{n\beta} - \vec{\chi}_\beta \times \overrightarrow{\Psi}_{n\alpha}\right) - \left[\vec{\chi}_\beta\left(\overrightarrow{\Psi}_{\alpha\alpha}\cdot\vec{\chi}_\beta\right) - \vec{\chi}_\alpha\left(\overrightarrow{\Psi}_{\beta\alpha}\cdot\vec{\chi}_\alpha\right)\right.$$

$$\left. + \vec{\chi}_\alpha\left(\overrightarrow{\Psi}_{\beta\beta}\cdot\vec{\chi}_\alpha\right) - \vec{\chi}_\beta\left(\overrightarrow{\Psi}_{\alpha\beta}\cdot\vec{\chi}_\alpha\right)\right] = \frac{\overrightarrow{T}\times\hat{n}}{\rho C_2^2}$$

$$\tag{32}$$

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The above four equations when combined with the limiting values
of Equations (6) and (7) constitute a system of eight simultaneous
equations for the determination of the eight surface potentials
ϕ , ψ , ϕ_n and ψ_n . If the stresses specified on the
boundary are time independent the problem is a static problem.
Consequently, the time derivative terms are not present in
Equations (31) and (32) when Equations (5) are used to eliminate
ϕ_{nn} and ψ_{nn} from the boundary conditions. These four equations,
together with the limiting forms of Equations (8) and (9), consti-
tute a system of eight equations for the required surface
potentials and their normal derivatives. Similarly, if a
steady harmonic time dependence is assumed, the terms ϕ_{tt} and
ψ_{tt} in Equations (31) and (32) are replaced by $-k_1^2 \phi$ and $-k_2^2 \psi$
respectively when the second normal derivatives are eliminated
from the boundary conditions by substitution from Equations (10).
Again, a set of determinate equations for the surface potentials
and their normal derivatives may be obtained by adjoining the
limiting values of the integral representations, Equations (11) and
(12). These sets of eight equations consitute the desired
integral equation formulation of the first fundamental boundary
value problem of elasticity.

The third fundamental problem of classical elasticity is
formulated in an analagous manner. The procedure is a direct
extension of the previous formulations. For the dynamic
problem the limiting values of Equations (6) and (7) must hold
everywhere on the boundary while for the static problem the
limiting values of Equations (8) and (9) must hold. The boundary
conditions require that on those sections of the boundary where

the surface displacements are prescribed, Equations (17) and
(18) are to be satisfied and on the remaining sections of σ
where the surface traction $\vec{T}$ is given, Equations (28) and (29)
must hold. These latter equations can be combined as before
with Equations (4) to eliminate ϕ_{nn} and $\vec{\psi}_{nn}$ for the dynamic
problem and combined with Equations (5) to eliminate the same
variables for the static problem. Consequently, at every point
of the boundary, there exists eight equations for the determination
of the eight surface potentials ϕ , $\vec{\psi}$, ϕ_n and $\vec{\psi}_n$. If the
dynamic problem is reduced to a harmonic time dependence, the
Helmholtz integral representations are to be used. The author
would like to point out that parts of those sections of Reference
(5) referring to the second and third fundamental problems are in
error and should be modified in accordance with the analysis
presented here.

For those problems in which the body force is absent the
displacement potentials satisfy homogeneous equations. Thus, in
the above formulations of the three fundamental problems the
volume integrals are not present in the integral equations.

Further Modifications and Considerations

When the elastic medium is a composite of two or more different elastic media the previous formulations must be modified. The procedure for setting up the required equations for two adjacent media will be outlined. The extension to more than two media may then be readily accomplished. Denote the media by medium I and medium II and let terms in medium I be denoted by the subscript I and those terms in medium II be denoted by the subscript II. In addition, let $\sigma_{I,II}$ denote that part of the interface between media I and II. Continuity of normal and tangential displacements across $\sigma_{I,II}$ implies

$$H(\phi_I)_n + \vec{\chi}_\beta \cdot (\vec{\Psi}_I)_\alpha - \vec{\chi}_\alpha \cdot (\vec{\Psi}_I)_\beta = H(\phi_{II})_n + \vec{\chi}_\beta \cdot (\vec{\Psi}_{II})_\alpha - \vec{\chi}_\alpha \cdot (\vec{\Psi}_{II})_\beta$$

(33)

and

$$H(\vec{\Psi}_I)_n + \vec{\chi}_\alpha \times (\vec{\Psi}_I)_\beta - \vec{\chi}_\beta \times (\vec{\Psi}_I)_\alpha + \vec{\chi}_\alpha (\phi_I)_\beta - \vec{\chi}_\beta (\phi_I)_\alpha$$

$$= H(\vec{\Psi}_{II})_n + \vec{\chi}_\alpha \times (\vec{\Psi}_{II})_\beta - \vec{\chi}_\beta \times (\vec{\Psi}_{II})_\alpha + \vec{\chi}_\alpha (\phi_{II})_\beta - \vec{\chi}_\beta (\phi_{II})_\alpha$$

(34)

Continuity of normal and tangential stresses across $\sigma_{I,II}$ implies

$$\rho_I \left\{ H^2 (\phi_I)_{nn} C_{D_I}^2 + (\phi_I)_{\alpha\alpha} \left(C_{D_I}^2 - 2 C_{S_I}^2 \right) G + (\phi_I)_{\beta\beta} \left(C_{D_I}^2 - 2 C_{S_I}^2 \right) E \right.$$

$$\left. + 2 C_{S_I}^2 H \left[\vec{\chi}_\beta \cdot (\vec{\Psi}_I)_{n\alpha} - \vec{\chi}_\alpha \cdot (\vec{\Psi}_I)_{n\beta} \right] \right\} = \rho_{II} \left\{ H^2 (\phi_{II})_{nn} C_{D_{II}}^2 \right.$$

$$+ (\phi_{II})_{\alpha\alpha} \left(C_{D_{II}}^2 - 2 C_{S_{II}}^2 \right) G + (\phi_{II})_{\beta\beta} \left(C_{D_{II}}^2 - 2 C_{S_{II}}^2 \right) E$$

$$\left. + 2 C_{S_{II}}^2 H \left[\vec{\chi}_\beta \cdot (\vec{\Psi}_{II})_{n\alpha} - \vec{\chi}_\alpha \cdot (\vec{\Psi}_{II})_{n\beta} \right] \right\}$$

(35)

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and

$$\rho_I C_{S_I}^2 \left\{ H^2 (\vec{\Psi}_I)_{nn} + 2H \left[\vec{\chi}_\alpha (\phi_I)_{n\beta} - \vec{\chi}_\beta (\phi_I)_{n\alpha} \right] - \left(\vec{\chi}_\beta \left[(\vec{\Psi}_I)_{\alpha\alpha} \cdot \vec{\chi}_\beta \right] \right. \right.$$

$$\left. - \vec{\chi}_\alpha \left[(\vec{\Psi}_I)_{\alpha\beta} \cdot \vec{\chi}_\beta \right] + \vec{\chi}_\alpha \left[(\vec{\Psi}_I)_{\beta\beta} \cdot \vec{\chi}_\alpha \right] - \vec{\chi}_\beta \left[(\vec{\Psi}_I)_{\beta\alpha} \cdot \vec{\chi}_\alpha \right] \right)$$

$$\left. + H \left[\vec{\chi}_\alpha \times (\vec{\Psi}_I)_{n\beta} - \vec{\chi}_\beta \times (\vec{\Psi}_I)_{n\alpha} \right] \right\} = \rho_{II} C_{S_{II}}^2 \left\{ H^2 (\vec{\Psi}_{II})_{nn} \right.$$

$$+ 2H \left[\vec{\chi}_\alpha (\phi_{II})_{n\beta} - \vec{\chi}_\beta (\phi_{II})_{n\alpha} \right] - \left(\vec{\chi}_\beta \left[(\vec{\Psi}_{II})_{\alpha\alpha} \cdot \vec{\chi}_\beta \right. \right.$$

$$\left. - \vec{\chi}_\alpha \left[(\vec{\Psi}_{II})_{\alpha\beta} \cdot \vec{\chi}_\beta \right] + \vec{\chi}_\alpha \left[(\vec{\Psi}_{II})_{\beta\beta} \cdot \vec{\chi}_\alpha \right] - \vec{\chi}_\beta \left[(\vec{\Psi}_{II})_{\beta\alpha} \cdot \vec{\chi}_\alpha \right] \right)$$

$$\left. + H \left[\vec{\chi}_\alpha \times (\vec{\Psi}_{II})_{n\beta} - \vec{\chi}_\beta \times (\vec{\Psi}_{II})_{n\alpha} \right] \right\}$$

(36)

where

$$C_{D_I}^2 = \frac{\lambda_I + 2\mu_I}{\rho_I} \qquad C_{D_{II}}^2 = \frac{\lambda_{II} + 2\mu_{II}}{\rho_{II}}$$

$$C_{S_I}^2 = \frac{\mu_I}{\rho_I} \qquad C_{S_{II}}^2 = \frac{\mu_{II}}{\rho_{II}}$$

The terms $(\phi_I)_{nn}$, $(\vec{\Psi}_I)_{nn}$, $(\phi_{II})_{nn}$ and $(\vec{\Psi}_{II})_{nn}$ may be eliminated
from Equations (35) and (36) by employing the displacement potential
equations appropriate for the problem and the medium. The
resulting eight equations when combined with the limiting values
of the integral representations yields sixteen equations for the
determination of the surface values of the sixteen functions
ϕ_I , $\vec{\Psi}_I$, ϕ_{II} , $\vec{\Psi}_{II}$, $(\phi_I)_n$, $(\vec{\Psi}_I)_n$, $(\phi_{II})_n$ and $(\vec{\Psi}_{II})_n$.

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In addition, on those segments of the boundary of either medium where the displacements or stresses are specified, the equations previously developed must hold. If the parameters α and β are so chosen as to be the arc lengths along the curves β = constant and α = constant respectively, the differential forms E, G and H are equal to unity. This simplifies the boundary conditions and considerably reduces the numerical effort required to obtain a solution to a specific problem.

In all of the previous development it was assumed that the elastic medium was finite and bounded by a closed surface. If the medium is infinite in extent, the integral representations must be suitably modified and the displacement potentials must vanish at infinity in a prescribed manner.[9],[12],[13] The existence of solutions for three-dimensional problems is discussed by Sokolnikoff and Gurtin, among others, who give several references.

Discussion and Conclusions

By employing methods analogous to those used in potential theory, the fundamental boundary value problems of static and dynamic elasticity have been formulated in terms of a coupled set of integral equations and inhomogeneous partial differential equations. One of the advantages of this formulation is that the shape of the boundary is directly accounted for. If the geometry of the cross-section coincides with a coordinate system in which the equations satisfied by the displacement potentials separate, it is possible to obtain solutions by eigenfunction techniques. Another advantage is that because of the similar structure of the formulation of each problem methods developed for the solution of one problem may more easily be adapted to obtain solutions for the others. For example, if a numerical method of solution is attempted, the integrands occuring in all of the formulations have the same type of singularity. Thus, a technique that satisfactorily estimates the contribution to the integral in the neighborhood of the singularity of one of the integrals may be used in estimating the contributions of the analogous singularities in the other integrals.

If in the static and harmonic time dependent problems the integrals are approximated by finite sums, the resultant set of linear equations is simultaneous. A similar numerical approximation applied to the integrals in the initial boundary value problem results in a set of linear equations which are not simultaneous. Hence, the implicit and explicit character of the two types of problems is clearly evident. Also, it should be noted that, the

first and second fundamental boundary value problems in dynamic elasticity are analogous to the problems of determining the scattered field due to an elastic wave impinging upon an arbitrary fixed or free surface.

After the desired surface potentials are determined, the displacement and stress fields are obtained by differentiating the kernels of the integrals and properly combining the resulting expressions with the known surface potentials and integrating. This method of computing the stress and displacement field avoids the decrease in rate of convergence of the infinite series that results when eigenfunction expansions are used to obtain these fields.

Bibliography

1. A. E. H. Love, "A Treatise on the Mathematical Theory of Elasticity," Dover Publications, New York, 1944.

2. N. I. Muskhelishvilli, "Some Basic Problems in the Mathematical Theory of Elasticity," translated by J. R. M. Radok, Erven P. Noordhoff, Groningen, Netherlands, 1953.

3. C. E. Pearson, "Theoretical Elasticity," Harvard University Press, Cambridge, Massachusetts, 1959.

4. I. S. Sokolnikoff, "Mathematical Theory of Elasticity," McGraw-Hill Book Company, Inc., New York, 1956.

5. R. P. Banaugh, "Scattering of Elastic Waves by Surfaces of Arbitrary Shape," University of California Lawrence Radiation Laboratory Report UCRL-6779, 1962.

6. P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," Volumes I and II, McGraw-Hill Book Company, Inc., New York, 1953.

7. I. N. Sneddon, "Elements of Partial Differential Equations," McGraw-Hill Book Company, Inc., New York, 1957.

8. W. D. MacMillan, "Theory of the Potential," McGraw-Hill Book Company, Inc., New York, 1930.

9. B. B. Baker and E. T. Copson, "The Mathematical Theory of Huygens' Principle," Oxford University-Clarendon Press, London, 1953, 2nd edition.

10. W. C. Graustein, "Differential Geometry," The Macmillan Company, New York, 1949.

11. S. P. Timoshenko and J. N. Goodier, "Theory of Elasticity," McGraw-Hill Book Company, Inc., New York, 1951.

12. M. E. Gurtin, "On Helmholtz's Theorem and the Completeness of the Papkovich-Neuber Stress Functions for Infinite Domains," ONR Report, Contract Number 562(25), NR-064-431.